Chapter 4.4 Kuratowski's theorem

Theorem - Kuratowski A graph G is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Let G be a graph. A graph H is a **minor** of G if H can be obtained from G by deleting vertices, deleting edges and contracting edges.

Theorem - Kuratowski A graph G is planar iff it does not contain K_5 or $K_{3,3}$ as a minor.

The main part of the proof is for 3-connected G. But first we add a lemma to be used in the proof.

Lemma If G is a 2-connected plane graph, then every face of G is bounded by a cycle.

1: Prove the Lemma. (Induction and ear decomposition)

Solution: The base of induction is G being a cycle, both faces are bounded by a cycle. Induction step: If G is not a cycle, there exists a path P such that $G = H + P$, where H is also 2-connected. Here H inherits the drawing from G. By induction, every face in H is bounded by a cycle. Not observe that adding P cuts one face into two faces and these faces are now also bounded by cycles.

Lemma Every 3-connected graph G without a K_5 or $K_{3,3}$ minor is planar.

Proof By induction on $|V(G)|$. Start with a 3-connected graph G without a K_5 or $K_{3,3}$ minor. If G is K_4, it is planar. Recall K_4 is smallest 3-connected graph.
Since G is 3-connected and not K_4, it contains edge xy such that G/xy is also 3-connected. Denote the contracted vertex by v_{xy}.

2: Show that G/xy does not contain K_5 or $K_{3,3}$ as a minor.

Solution: If it contained one of the minors, G would also contain them. The vertex v_{xy} would be simply replaced by both x and y.

Now we use induction to draw G/xy. Let G' be obtained from the drawing of G/xy when we remove v_{xy}. Notice it is still a plane drawing.

3: Use G/xy and G' to obtain a drawing of $G - y$.

Solution: Use the drawing of G/xy and remove edges $v_{xy}z$ if xz is not an edge in G. Then map x to the same point in the plane as v_{xy}. Notice that $G - y$ is a subgraph of G/xy which makes it easy to inherit the drawing.

Let C be the cycle bounding face of $G/xy - v_{xy}$. (Why such cycle exists?) Now all neighbors of x are on the cycle C. Denote them by $x_1, \ldots, x_k$. Denote by P_i a subpath of C starting in x_i, ending in x_{i+1}, and not containing any other x_j. Here we use $1 = k + 1$, i.e., counting $\mod k$.

4: Assume we are lucky and all neighbors of y are in some P_i. Finish the proof by finding a drawing of G in this case.

Solution: There is a cycle $P_i + x$ that bounds a face, call it f. Now one can draw y in the face and connect it to $N(y)$ without any crossings. Proving this formally is a little harder.

5: Assume y and x have three common neighbors on C. Finish the proof by finding some contradiction.

Solution: If they have three common neighbors, then there is a topological minor of K_5. And that is a contradiction.

6: Show that y has to neighbors y' and y'' on C such that y' and y'' are separated in C by x' and x'', which are neighbors of x. Finish the proof by finding a contradiction.

Solution: If y has a neighbor y' outside of $N(x)$ in P_i, it has another neighbor y'' not in P_i. The endpoints of P_i are x' and x''. Otherwise y has exactly 2 neighbors y and y' in C that are also neighbors of x. Since y' and y'' are not in the same P_i, there are x' and x'' neighbors of x separating them. x, y, x', x'', y', y'' form $TK_{3,3}$.

All that remains is to show the assumption for 3-connected graphs is easy to satisfy.

Let G be a graph without TK_5 or $TK_{3,3}$. Add edges to G as long as there is no TK_5 or $TK_{3,3}$. G is then edge-maximal.

We will use the following lemma with $\mathcal{X} = \{K_5, K_{3,3}\}$.

Lemma Let $\mathcal{X}$ be a set of 3-connected graphs. Let G be an edge maximal-graph without a topological minor in $\mathcal{X}$. If there is a separator S of order at most 2 of G, then $|S| = 2$ and $G[S] = K_2$. Moreover, if the $V_1, V_2 \subseteq V(G)$ are the separation, i.e., $V_1 \cap V_2 = S$ and there are no edges between $V_1 \setminus S$ and $V_2 \setminus S$, then $G[V_1]$ and $G[V_2]$ are also edge-maximal without a topological minor in $\mathcal{X}$.

Proof idea Graphs in $\mathcal{X}$ have minimum degree at least 3. Hence all branch vertices (not on subdivided edges) must be all in V_1 or V_2. The rest of the proof is testing few cases.

Lemma If G has at least 4 vertices and G is edge-maximal without TK_5, $TK_{3,3}$, then G is 3-connected.

Proof By induction on the number of vertices of G.

7: Finish the proof

Solution: If G is 3-connected, we are done. So it is not 3-connected. By Lemma above, it has a 2-cut xy that is an edge. Spit G along xy into two smaller graphs G_1 and G_2 such that their intersection is xy. By the previous Lemma, both G_1 and G_2 are 3-connected. They have no TK_5 or $TK_{3,3}$. So we can draw both of them. We can draw them with xy being on the outer face f. Now we can 'move the drawings' such that the xy edge coincides and we can add one more edge between vertices on f. It eliminates does not create TK_5 or $TK_{3,3}$ since the drawing is planar. Contradiction with maximality of number of edges.

Note that the book has a slightly different proof that is more convincing in the drawing sense. Here we are moving the drawings to match xy edge, which the book does not do. But it takes longer.